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On Rational Series in One Variable over certain Dioids

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Abstract: We give a characterization of rational series in one variable over certain idempotent semirings (commutative dioids) such as for instance the "$(\max, +)$ "semiring. We show that a series is rational iff it is merge of ultimately geometric series. As a by-product, we obtain a new proof of the periodicity theorem for powers of irreducible matrices and also some more general auxiliary results. We apply this characterization of rational series to the minimal realization problem for which we obtain an upper bound. We also obtain a lower bound in terms of minors in a symmetrized semiring.

Key-words: Rational series, Dioids, Max-Plus algebra, Minimal realization, Discrete Event Systems

(Résumé : tsvp)

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Sur les séries rationnelles en une indéterminée à coefficients dans certains dioïdes

Résumé : Nous caractérisons les séries rationnelles en une indéterminée sur certains semianneaux idempotents (dioïdes commutatifs), par exemple le semianneau "$(\max, +)$". Nous montrons que les séries rationnelles sont obtenues par emboîtement de séries ultimement géométriques. On obtient comme conséquence une nouvelle preuve du théorème de périodicité des puissances de matrices irréductibles dans ces semianneaux, ainsi que des variantes plus générales. Comme autre application, on donne une borne supérieure pour la dimension minimale de réalisation. Une borne inférieure fait intervenir les mineurs dans un semianneau symmétrisé.

Mots-clé : Séries rationnelles, Dioïdes, Algèbre $(\max, +)$, Réalisation minimale, Systèmes à Événements Discrets

Introduction

The traditional term "(max, +) algebra" refers to the semiring $\mathbb{R}_{\max} \stackrel{\text{def}}{=} (\mathbb{R} \cup \{-\infty\}, \max, +)$, that is to the set $\mathbb{R} \cup \{-\infty\}$ equipped with max as addition, denoted by $\oplus$ (e.g. $2 \oplus 3 = 3$) and $+$ as product (denoted by $\otimes$, e.g. $1 \otimes 2 = 3$). Some specific notation for the neutral elements is also useful: $\varepsilon \stackrel{\text{def}}{=} -\infty$ denotes the *zero*, such that $x \oplus \varepsilon = x$, $x \otimes \varepsilon = \varepsilon$ and $e \stackrel{\text{def}}{=} 0$ denotes the *unit* (such that $e \otimes x = x$). This algebraic structure has been widely studied [9, 20, 1, 27]. It is known [31, 1] that an interesting subclass of Discrete Event Systems consists in causal (max, +) linear stationary operators, which can be represented by *convolutions* over the (max, +) algebra, that is

$$u \mapsto y = h * u : \qquad y(n) = h * u(n) \stackrel{\text{def}}{=} \bigoplus_{k \in \mathbb{N}} h(k) \otimes u(n-k) = \sup_{k \in \mathbb{N}} [h(k) + u(n-k)] \tag{1}$$

(u, y are maps $\mathbb{Z} \to \mathbb{R}_{\max}$, $h : \mathbb{N} \to \mathbb{R}_{\max}$). A case of particular interest arises when the *transfer* h admit a finite dimensional linear representation, that is, when there exists an integer p and three matrices $A \in \mathbb{R}_{\max}^{p \times p}, B \in \mathbb{R}_{\max}^{p \times 1}, C \in \mathbb{R}_{\max}^{1 \times p}$ such that

$$y(k) = \bigoplus_{k \in \mathbb{N}} C A^k B u(n-k) \ . \tag{2}$$

Here, as usual, concatenation denotes the matrix product induced by the semiring structure, that is

$$(UV)_{ij} \stackrel{\text{def}}{=} (U \otimes V)_{ij} \stackrel{\text{def}}{=} \bigoplus_k U_{ik} \otimes V_{kj}$$

and consequently, A^k stands for $A \otimes \ldots \otimes A$ (k times). This leads us to introducing the semiring $\mathbb{R}_{\max}[[X]]$ of formal series in a single indeterminate X over $\mathbb{R}_{\max}$: the input-output behavior of the system is determined by its *transfer series*

$$H = \bigoplus_{k \in \mathbb{N}} C A^k B \, X^k \in \mathbb{R}_{\max}[[X]] \ . \tag{3}$$

This provides a motivation for the study of *realizable* series, that is, of series H for which there exists a finite dimensional triple (A, B, C) providing a representation of the form (3). Realizable series are classically related with *rational* series that we next define after some notation. Given a series H, we shall write $\langle H, X^k \rangle$ or H_k for the coefficient of H at X^k, so that H writes

$$H = \bigoplus_{k \in \mathbb{N}} H_k \, X^k = \bigoplus_{k \in \mathbb{N}} \langle H, X^k \rangle X^k \ .$$

Now, let $s \in \mathbb{R}_{\max}[[X]]$ with zero constant coefficient, i.e. $\langle s, X^0 \rangle = \varepsilon$. The star of s which can be written formally

$$s^* = \bigoplus_{i \in \mathbb{N}} s^i$$

is by definition the unique series such that

$$\forall k, \ \langle s^*, X^k \rangle = \bigoplus_{i \in \mathbb{N}} \langle s^i, X^k \rangle$$

(this is indeed a finite sum: $\langle s^i, X^k \rangle = \varepsilon$ for i large enough due to the fact that s has zero constant coefficient). Since series over the (max, +) algebra are naturally ordered, it is possible to define alternatively s^* as the least upper bound of the infinite set $\{s^0, s^1, \ldots\}$, when it exists. We do not adopt this alternative convention here, but it leads to the same class of series (this in discussed in detail in the Appendix II).

0.0.1 Definition *The semiring* Rat *of rational series is the least set of formal series containing polynomials and such that*

$$\begin{array}{lcl} \mathrm{Rat} \oplus \mathrm{Rat} & \subset & \mathrm{Rat} \\ \mathrm{Rat} \otimes \mathrm{Rat} & \subset & \mathrm{Rat} \\ \mathrm{Rat}^* & \subset & \mathrm{Rat} \end{array}$$

Since s^* is only defined for s with zero constant coefficient, $\mathrm{Rat}^* \subset \mathrm{Rat}$ indeed means that

$$\text{(S)} \quad \text{if } s \in \mathrm{Rat} \text{ and } \langle s, X^0 \rangle = \varepsilon, \text{ then } s^* \in \mathrm{Rat}.$$

The celebrated Kleene-Schützenberger theorem [3] states that a series is rational iff it is realizable. Therefore, the theory of systems of type (2) leads us to studying $(\max, +)$ rational series.

In this paper, we present some characterizations of rational series over the $(\max, +)$ algebra and others idempotent semirings (dioids). As a by-product, we obtain some rather general cyclicity theorems for powers of matrices. Then, we present some bounds for the minimal realization problem. Most of these results are taken from the thesis of the author [16], with some generalizations. The characterization of rational series holds under a "weak stabilization" condition analogous to the stabilization condition already introduced by Dudnikov and Samborskiĭ. Weak stabilization requires the sum of two geometric sequences to be ultimately geometric. Then, rational series are obtained by merging some ultimately geometric series. This is a bit similar to the case of rational series over $(\mathbb{R}^+, +, \times)$ (see [3], Chapter 5) and extends some results given by Moller [29] for rational series over $\mathbb{R}_{\max}$ and by Cohen,Moller,Quadrat and Viot [8] for rational series in the dioid of shift operators in timed event graphs (called $\mathcal{M}_{\mathrm{in}}^{\mathrm{ax}}[[\gamma, \delta]]$). Some related results motivated by logical problems have been obtained by Bonnier and Krob [24] for rational series over the *tropical* semiring (i.e. $(\mathbb{N} \cup \{+\infty\}, \min, +)$). As an application, we obtain with these techniques some cyclicity theorem in rather general dioids. This is because there is a natural connection between rational series and sequences of powers of matrices (via generating series). Thus, the representation theorems proved for rational series automatically transfer to asymptotic theorems for matrices. In particular, we obtain another proof –with some relaxed assumptions– of the cyclicity theorem [7, 1, 12], which states that an irreducible matrix A is projectively torsion, that is $A^{n+c} = \lambda A^n$ for some integers $n \geq 0, c \geq 1$ and scalar λ. Next, we provide two bounds for the minimal realization problem: it consists in minimizing the size p of the linear representation of a rational series H (p is the size of A in (3)). The upper bound is a simple extension of a result of Cuninhame-Green that we mention here for the sake of completeness. The second one relies on a combinatorial identity (Binet-Cauchy formula) valid in a *symmetrized* semiring of $\mathbb{R}_{\max}$. Indeed, the bound is more general since it is valid in any semiring (in the case of fields, it coincides with the traditional characterization in terms of the minors of the Hankel matrix [14]). We illustrate these two bounds by giving a few examples: we provide some equality cases, but we also show that the bounds can be arbitrarily coarse. We conclude the paper by showing how determinants can be computed in the $(\max, +)$ algebra in order to make the minor bound effective. Let us also mention that rational series with several noncommuting indeterminates over similar dioids have been considered by Hashiguchi, Simon, Krob, Mascle,Leung [21, 33, 23, 26, 25] and also by the author in [18, 17].

1 Rational Series over Commutative Dioids

1.1 General Characterizations

We first characterize rational series over commutative *dioids* [1] (a dioid is a semiring whose addition is idempotent, i.e. $a \oplus a = a$). Of course, the main dioid that we have in mind for applications is $\mathbb{R}_{\max}$, but the result that we give is more general.

1.1.1 Definition *Let $\mathcal{S}$ denote a commutative semiring. We say that $s \in \mathcal{S}[[X]]$ is* ultimately geometric *if $\exists N \in \mathbb{N}, c \in \mathbb{N}\backslash\{0\}, \lambda \in \mathbb{R}_{\max}$ such that*

$$n \geq N \Rightarrow \langle s, X^{n+c} \rangle = \lambda \langle s, X^n \rangle \ . \tag{4}$$

This property admits an immediate algebraic characterization:

1.1.2 Proposition *Let $\mathcal{S}$ be a commutative semiring. The series $s \in \mathcal{S}[[X]]$ is* ultimately geometric *iff there exists two polynomials $p, q \in \mathcal{S}[X]$, $c \in \mathbb{N}\backslash\{0\}$,$\lambda \in \mathcal{S}$ such that*

$$s = p \oplus q(\lambda X^c)^* \tag{5}$$

Proof (4) $\Rightarrow$ (5). This follows from

$$s = \bigoplus_{n \leq N-1} \langle s, X^n \rangle X^n \oplus \left(\bigoplus_{n=N}^{N+c-1} \langle s, X^n \rangle \right) (\lambda X^c)^* \ .$$

(5) $\Rightarrow$ (4). We have, for $n > \deg p$,

$$\begin{aligned}
\langle s, X^{n+c} \rangle &= \langle q(\lambda X^c)^*, X^{n+c} \rangle \\
&= \bigoplus_{k=0}^{\deg q} \langle q, X^k \rangle \langle (\lambda X^c)^*, X^{n+c-k} \rangle \\
&= \bigoplus_{k=0}^{\deg q} \langle q, X^k \rangle \lambda \langle (\lambda X^c)^*, X^{n-k} \rangle \\
&= \lambda \langle s, X^n \rangle \ .
\end{aligned}$$

■

Consider $\mathcal{S} = \mathbb{B} = \{\varepsilon, e\}$ (the boolean semiring). A series $s \in \mathbb{B}[[X]]$ is rational iff its support

$$\operatorname{supp} s \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid \ \langle s, X^n \rangle \neq \varepsilon\}$$

is a rational subset of $\mathbb{N}$. It is well known ([13], Chapter V, Proposition 1.1) that a subset of $\mathbb{N}$ is rational iff it is the union of a finite set and of a finite number of arithmetic progressions. The following fact, already noticed by Moller in the case of $\mathbb{R}_{\max}$ ([29], 7.13.4), extends this property.

1.1.3 Theorem *Let $\mathcal{S}$ be a commutative dioid. A series $s \in \mathcal{S}[[X]]$ is rational if and only if it is a finite sum of ultimately geometric series.*

Proof Let $\mathcal{G}$ denote the set of finite sums of series of the form (5) and $\mathcal{R}$ denote the set of rational series. Since $\mathcal{S}[X] \subset \mathcal{G} \subset \mathcal{R}$, it is enough to show that $\mathcal{G}$ is closed under addition (this is obvious), product, and star. Let $a, b \in \mathcal{S}[[X]]$ such that $\langle a, X^0\rangle = \langle b, X^0\rangle = \varepsilon$. The following well known rational identities [24, 8, 16] hold in commutative dioids

$$
\begin{aligned}
(a \oplus b)^* &= a^* b^* && (6)\\
(ba^*)^* &= e \oplus b(a \oplus b)^* && (7)\\
\forall k \geq 1,\ a^* &= (e \oplus a \oplus \ldots \oplus a^{k-1})(a^k)^* \ . && (8)
\end{aligned}
$$

In particular, for $a = \lambda X^c$ (with $\lambda \in \mathcal{S}$, $c \geq 1$),

$$
(b(\lambda X^c)^*)^* = e \oplus bb^*(\lambda X^c)^* \ . \tag{9}
$$

Hence, $\mathcal{G} \otimes \mathcal{G} \subset \mathcal{G} \Rightarrow \mathcal{G}^* \subset \mathcal{G}$. It remains to show that $\mathcal{G} \otimes \mathcal{G} \subset \mathcal{G}$. Indeed, it is enough to check that for all $c, d \in \mathbb{N}\backslash\{0\}$, $\lambda, \mu \in \mathcal{S}$, we have

$$
t \stackrel{\text{def}}{=} (\lambda X^c)^*(\mu X^d)^* \in \mathcal{G} \ . \tag{10}
$$

Define c', d' by $cc' = dd' = \mathrm{lcm}(c, d)$. Then, the assertion (10) follows from

$$
t = (\bigoplus_{i=0}^{c'-1} \lambda^i X^{ci})(\lambda^{c'} X^{\mathrm{lcm}(c,d)})^* (\bigoplus_{j=0}^{d'-1} \mu^j X^{dj})(\mu^{d'} X^{\mathrm{lcm}(c,d)})^* \tag{11}
$$

(by (8)) together with

$$
(\lambda^{c'} X^{\mathrm{lcm}(c,d)})^* (\mu^{d'} X^{\mathrm{lcm}(c,d)})^* = ((\lambda^{c'} \oplus \mu^{d'}) X^{\mathrm{lcm}(c,d)})^*
$$

(by (6)). ■

In order to characterize rational series, we introduce the following notion, which already appears in the theory of rational positive series [3]. The merge of k series $s^{(0)}, \ldots, s^{(k)}$ is obtained by taking alternatively the coefficients of these series. More precisely:

1.1.4 Definition (Merge of Series) *We say that s is the* merge *of the series $s^{(0)}, \ldots, s^{(k-1)} \in \mathcal{S}[[X]]$ if*

$$
\forall i \in \{0, \ldots, k-1\},\ \forall n \geq 0,\ \ \langle s, X^{nk+i}\rangle = \langle s^{(i)}, X^n\rangle \ .
$$

For instance, the series

$$
\langle s, X^{2p}\rangle = 2p,\ \langle s, X^{2p+1}\rangle = e
$$

is the merge of $(2X)^*$ and X^*. More generally, it is easily realized that a series s is the merge of ultimately geometric series iff the following periodicity property holds

$$
\begin{aligned}
\exists N \in \mathbb{N}, \exists c \in \mathbb{N}\backslash\{0\}, \exists \lambda_0, \ldots, \lambda_{c-1} \in \mathbb{R}_{\max},\quad & \forall i \in \{0, \ldots, c-1\}, \forall n \geq N,\\
& \langle s, X^{nc+i+c}\rangle = \lambda_i \langle s, X^{nc+i}\rangle \ .
\end{aligned} \tag{12}
$$

We shall also need the two following conditions (the second one is borrowed to Dudnikov and Samborskiĭ, [11], Condition 2.2).

1.1.5 Definition (Weak and Strong Stabilization) *The dioid $\mathcal{S}$ satisfies the* weak stabilization condition *if for all $a, b, \lambda, \mu \in \mathcal{S}$, there exists $c, \nu \in \mathcal{S}$ and $N \in \mathbb{N}$ such that*

$$
n \geq N \Rightarrow a\lambda^n \oplus b\mu^n = c\nu^n \ .
$$

If this property holds with $\nu = \lambda \oplus \mu$ (as soon as $a, b \neq \varepsilon$), we say that $\mathcal{S}$ satisfies the strong stabilization condition.

The scope of these conditions should become clear after a few examples.

1.1.6 Example The following dioids satisfy the strong stabilization condition:

1. $\mathbb{R}_{\max}$ and its subdioids (e.g. $(\mathbb{N}\cup\{-\infty\},\max,+)$).
2. The completed dioid of $\mathbb{R}_{\max}$, i.e. $\overline{\mathbb{R}}_{\max} \stackrel{\text{def}}{=} (\mathbb{R}\cup\{\pm\infty\},\max,+)$.
3. $(\mathbb{R}^n\cup\{\varepsilon\},\max,+)$ (obtained by adjoining a zero element to $(\mathbb{R}^n,\max,+)$, i.e. $\varepsilon\oplus x \stackrel{\text{def}}{=} x, \varepsilon\otimes x \stackrel{\text{def}}{=} x$)
4. $(\mathbb{N}^*\cup\{\varepsilon\},\text{lcm},\times)$, where ε is also a zero and lcm is seen as a binary law (e.g. $6\oplus 4 = \text{lcm}(6,4) = 12$).
5. The set of (not necessarily bounded) closed intervals of $\mathbb{R}$, equipped with $A\oplus B = \text{conv}(A\cup B)$, $A\otimes B = A+B$ (conv denotes the convex hull, $+$ the vector sum).
6. Let $(\mathcal{D},\vee,\wedge)$ be any distributive lattice with universal lower bound $\perp$ and upper bound $\top$ (i.e. $\perp\vee x = x, \top\wedge x = x, \forall x$). Then, setting $\oplus \stackrel{\text{def}}{=} \vee$ and $\otimes \stackrel{\text{def}}{=} \wedge$, $\mathcal{D}$ becomes a dioid which satisfies trivially the strong stabilization condition, for $\lambda^n = \lambda\wedge\ldots\wedge\lambda = \lambda$ for $n\geq 1$ and
$$a\lambda^n \oplus b\mu^n = (a\wedge\lambda)\vee(b\wedge\mu) = c\wedge(\lambda\vee\mu) = c(\lambda\oplus\mu)^n = \text{cst for } n\geq 1,$$
with $c \stackrel{\text{def}}{=} (a\vee b)\wedge(a\vee\mu)\wedge(\lambda\vee b)$ due to the distributivity.

Given a dioid $\mathcal{D}$, let $\mathcal{D}^n_c$ denote the set $\mathcal{D}^n$ equipped with componentwise[1] sum and product ($(u\oplus v)_i \stackrel{\text{def}}{=} u_i\oplus v_i, (u\otimes v)_i \stackrel{\text{def}}{=} u_i\otimes v_i$). The following obvious fact is worth noticing:

1.1.7 Observation *If $\mathcal{D}$ satisfies the weak stabilization condition, then so does $\mathcal{D}^n_c$.*

1.1.8 Example For $n\geq 2$, ${\mathbb{R}_{\max}}^n_c = ((\mathbb{R}\cup\{-\infty\})^n,\max,+)$ satisfies the weak stabilization condition (by the preceding proposition) but not the strong one. For instance, for $n=2$, let $a=\lambda=(e,e)$, $b=(e,\varepsilon)$, $\mu=(\varepsilon,1)$. We have
$$a\lambda^n\oplus b\mu^n = (e,e)^n\oplus(e,\varepsilon)(\varepsilon,1)^n = (e,e),\quad \forall n\enspace .$$
There does not exist c such that this sum of geometric series reduces to $c(\lambda\oplus\mu)^n = c\otimes(e,n)$.

1.1.9 Example The weak stabilization condition fails for the following dioids:

1. The set of finite subsets of $\mathbb{N}$, equipped with $\cup$ as addition and $+$ as product (consider $\{0\}^n\oplus\{1\}^n$).
2. The set of compact convex subsets of $\mathbb{R}^k$ (with $k\geq 2$), equipped with $a\oplus b \stackrel{\text{def}}{=} \text{conv}(a\cup b)$, $a\otimes b = a+b$ (let $o \stackrel{\text{def}}{=} \{(0,0)\}, i \stackrel{\text{def}}{=} \{(1,0)\}, j \stackrel{\text{def}}{=} \{(0,1)\}\subset\mathbb{R}^2$, and consider $o^n\oplus j(o\oplus i)^n$).
3. $\mathbb{S}_{\max}$, the symmetrized semiring of $\mathbb{R}_{\max}$ [30, 16] (consider $e^n\oplus(\ominus e)^n$).

We are now in position to state the main theorem which extends a periodicity result given by Cohen,Moller,Quadrat and Viot [8] (Theorem 21) for a particular dioid of shift operators. The periodicity theorem of [8] is essentially equivalent to our periodicity result restricted to the subclass of nondecreasing series in $\mathbb{R}_{\max}[[X]]$ (a series is nondecreasing if $n\leq m\Rightarrow\langle s,X^n\rangle\leq\langle s,X^m\rangle$).

[1] We use the notation $\mathcal{D}^n_c$ in order to distinguish $\mathcal{D}^2_c$ from the free symmetrized dioid $\mathcal{D}^2$ to be defined in §2.2.

1.1.10 Theorem *Let $\mathcal{S}$ be a commutative dioid which satisfies the weak stabilization condition. Then a series $s \in \mathcal{S}[[X]]$ is rational iff it is a merge of ultimately geometric series.*

Proof 1/ The merge of the rational series $s^{(0)}, \ldots, s^{(k-1)}$ is clearly equal to

$$s = \bigoplus_{i=0}^{k-1} X^i s^{(i)}(X^k) \tag{13}$$

where $s^{(i)}(X^k)$ denotes the series obtained by substitution of X^k to X in $s^{(i)}$. Hence it is rational.
2/ Conversely, let s be a rational series. It follows from Theorem 1.1.3 that s is a finite sum

$$s = p \oplus \bigoplus_{i=1}^{r} q_i(\lambda_i X^{c_i})^* \ . \tag{14}$$

After replacing c_i by $\mathrm{lcm}(c_1, \ldots, c_m)$ as in (11), we may assume that $c_i = c$ (constant). Then, it is enough to show that the series

$$s^{(j)} \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{N}} \langle s, X^{nc+j}\rangle X^n, \quad 0 \le j \le c-1$$

are ultimately geometric, since s is the merge of $s^{(0)}, \ldots s^{(c-1)}$. Clearly, we have from (14)

$$s^{(j)} = p'_j \oplus \bigoplus_{i=1}^{r} q'_{ij}(\lambda_i X)^* \ . \tag{15}$$

for some polynomials p'_j, q'_{ij}. We show that such a sum is ultimately geometric. Consider for $a, b \in \mathbb{N}$,$\alpha, \beta, \lambda \neq \mu \in \mathcal{S}$:

$$t = \alpha X^a(\lambda X)^* \oplus \beta X^b(\mu X)^* \ . \tag{16}$$

Then, for $n \ge \max(a, b)$, we have

$$\langle t, X^n\rangle = \alpha\lambda^{n-a} \oplus \beta\mu^{n-b} = (\alpha\lambda^{\max(a,b)-a})\lambda^{n-\max(a,b)} \oplus (\beta\mu^{\max(a,b)-b})\mu^{n-\max(a,b)}$$

The weak stabilization condition implies that there exists $N \in \mathbb{N}$ and $c, \nu \in \mathcal{S}$ such that

$$n - \max(a, b) \ge N \Rightarrow \langle t, X^n\rangle = c\nu^{n-\max(a,b)} \ .$$

Thus, t is ultimately geometric and the proof of Proposition 1.1.2 yields

$$t = u \oplus m(\nu X)^* \tag{17}$$

where u is a polynomial of degree $< N + \max(a, b)$ and $m = c\nu^N X^{N+\max(a,b)}$. Applying inductively the rewriting rule (16)→(17) to the sum (15), we get $s^{(j)} = v \oplus w(\nu' X)^*$, where v is a polynomial, w a monomial, and $\nu' \in \mathcal{S}$. By Proposition 1.1.2, this shows that $s^{(j)}$ is ultimately geometric. ■

1.2 Application to Cyclicity Theorems for Powers of Matrices

Since there is a close connection between sequences of powers of matrices and rational series, the above periodicity results given for rational series automatically provide some periodicity results for matrices. We first state the most general one. For a diagonalizable matrix A (over a field) with eigenvalues $\lambda_1, \ldots, \lambda_k$, we have obviously

$$\forall ij,\ \exists \alpha_1, \ldots, \alpha_k, \quad \forall n \in \mathbb{N}, \qquad A^n_{ij} = \sum_{r=1}^{k} \alpha_r \lambda_r^n \ . \tag{18}$$

An analogous property holds in general commutative dioids, but we have to take into account the cyclicity and the transient behavior.

1.2.1 Theorem *Let $\mathcal{S}$ be an arbitrary commutative dioid and $A \in \mathcal{S}^{p\times p}$. Then, there exists $c \geq 1$, $N \in \mathbb{N}$ such that for all ij, for all $l \in \{0,\ldots,c-1\}$, there exists a finite family of scalars $\alpha_1,\lambda_1,\ldots,\alpha_k,\lambda_k$ such that*

$$\forall n \geq N, \qquad A_{ij}^{nc+l} = \bigoplus_{r=1}^{k} \alpha_r \lambda_r^{n-N} \ . \tag{19}$$

In other words, the sequence $\{A_{ij}^n\}_{n\in\mathbb{N}}$ ultimately coincides with a merge of sums of geometric series. When $c=1$ and $N=0$, (19) reduces to the familiar (18).

Proof The sequence $A_{ij}^n = CA^nB$ where $B_k = \delta_{kj}$,$C_k = \delta_{ki}$ (Kroneker's delta) is realizable, hence by the Kleene-Schützenberger theorem [3], the "generating series"

$$s_{ij} = \bigoplus_n A_{ij}^n X^n = \bigoplus_n CA^nBX^n = (AX)_{ij}^* \in \mathcal{S}[[X]]$$

is rational. An application of Theorem 1.1.3 gives

$$s_{ij} = P \oplus \bigoplus_{r=1}^{k} Q_r(\lambda_r X^{c_r})^* \ . \tag{20}$$

where P and Q_k are polynomials and λ_k are scalars. Perhaps after changing the Q_k according to (8), we may assume that $c_k = c = \mathrm{cst}$ (and even that this constant does not depend on ij). Then, the property follows from

$$\langle Q_k(\lambda_k X^c)^*, X^{l+nc}\rangle = \bigoplus_{\substack{m\equiv l \pmod c\\ m\leq \deg Q_k}} \langle Q_k, X^m\rangle \langle(\lambda_k X^c)^*, X^{l+nc-m}\rangle = \bigoplus_{\substack{m\equiv l \pmod c\\ m\leq \deg Q_k}} \langle Q_k, X^m\rangle \lambda_k^{n+(l-m)/c} \ ,$$

which is meaningful as soon as

$$\forall m \equiv l \pmod c,\ m \leq \deg Q_k \Rightarrow n + (l-m)/c \geq 0 \ .$$

Thus (19) holds for all N such that

$$cN \geq \max(\deg P + 1, \max_k \deg Q_k) \ .$$

■

Under weak stabilization, we have the following more precise property:

1.2.2 Proposition *Let $\mathcal{S}$ be a commutative dioid satisfying the weak stabilization condition and let $A \in \mathcal{S}^{p\times p}$. For all $i,j \in \{1,\ldots,p\}$, there exists $c \in \mathbb{N}\setminus\{0\}$, $\lambda_0,\ldots,\lambda_{c-1} \in \mathcal{S}$, $N \in \mathbb{N}$, such that $\forall l \in \{0,\ldots,c-1\}$,*

$$n \geq N \Rightarrow \quad A_{ij}^{n+l+c} = \lambda_l A_{ij}^{n+l} \ . \tag{21}$$

Proof As shown in the proof of the preceding theorem, the sequence $A_{ij}^n = CA^nB$ is rational. It remains to apply Theorem 1.1.10 together with Property (12). ■

This extends the well known periodicity property for irreducible matrices in the $(\max,+)$-algebra (see e.g. [1]) if A is *irreducible*, there exists c, λ such that

$$A^{n+c} = \lambda^c A^n \tag{22}$$

for n large enough. Dudnikov and Samborskiĭ[12, 11] have proved that this result holds in dioids with cancellative product which satisfy the (strong) stabilization condition. Rational series provide an alternative proof under extended assumptions. Let us recall that a semiring has *no divisors of zero* if

$$ab = \varepsilon \Rightarrow a = \varepsilon \text{ or } b = \varepsilon$$

(in the case of dioids, this property is weaker than cancellativity).

1.2.3 Theorem (Cyclicity) *Let $\mathcal{S}$ be a commutative dioid without divisors of zero and satisfying the strong stabilization condition. If A is* irreducible*, then there exists $\lambda \in \mathcal{S}$, $c \geq 1, N \in \mathbb{N}$ such that*

$$n \geq N \Rightarrow A^{n+c} = \lambda A^n \enspace .$$

Proof From Theorem 1.1.3, we have for each ij a rational expression of the form

$$(AX)^*_{ij} = \bigoplus_{k \in F_{ij}} u_{kij} X^{l_{kij}} (\lambda_{kij} X^c)^* \qquad \lambda_{ijk}, u_{ijk} \in \mathcal{S} \enspace , \tag{23}$$

where c can chosen independent of ij (by (8)) and where for all ij, F_{ij} is a finite set. By Proposition 1.1.2, it is enough to show that up to a polynomial term, it is possible to take $\lambda_{ijk} = \lambda$ (independent of ijk) in (23).

The following Lemma is central:

1.2.4 Lemma *Assume that the strong stabilization condition holds. Let $m, n \in \mathbb{N}$, $p \geq 1$, $a, b \in \mathcal{S} \setminus \{\varepsilon\}, \lambda, \mu \in \mathcal{S}$. If $m \equiv n \pmod p$, there exists a polynomial P, $c \in \mathcal{S}$ and an integer $q \equiv m \pmod p$ such that*

$$aX^m (\lambda X^p)^* \oplus bX^n (\mu X^p)^* = P \oplus cX^q ((\lambda \oplus \mu) X^p)^* \enspace . \tag{24}$$

Proof of the Lemma. We shall use the following obvious identity:

$$\forall r \geq 1, \;\; \alpha^* = e \oplus \alpha \oplus \ldots \oplus \alpha^{r-1} \oplus \alpha^r \alpha^* \enspace . \tag{25}$$

Let us assume for instance that $m \geq n$. Then, there exists r such that $n + rp = m$, and applying (25) to $\alpha = \mu X^p$, we get $bX^n(\mu X^p)^* = P \oplus b\mu^r X^m (\mu X^p)^*$ for some polynomial P. Since the strong stabilization implies that

$$aX^m (\lambda X^p)^* \oplus b\mu^r X^m (\mu X^p)^* = cX^m ((\lambda \oplus \mu) X^p)^*$$

for a certain c, the Lemma is proved. ■

We return to the proof of the Theorem.

1/ *We first assume that A is primitive*, i.e. as in the Perron-Frobenius theory [2, 28] that all the entries of A^K are different from ε for a certain K. A fortiori, it is possible to choose k satisfying the following condition:

$$\forall l \in \{0, \ldots, c-1\}, \;\; \forall i, j, \;\; \exists m \leq k, \; m \equiv l \pmod c \text{ and } A^m_{ij} \neq \varepsilon \enspace . \tag{26}$$

The graphical interpretation [28, 1] should make this statement intuitive: due to the primitivity, there is a path of length at most k and of arbitrary remainder modulo c between any nodes of the graph associated with A.

The identity

$$(AX)^*_{ij} = \bigoplus_{\substack{0\le m\le k\\ 1\le u\le \dim A}} (AX)^m_{iu}(AX)^*_{uj} \ . \tag{27}$$

is the projection on the entry (ij) of the following obvious rational identity:

$$(AX)^* = (\mathrm{Id} \oplus AX \oplus \ldots \oplus (AX)^{k-1})(AX)^* \ . \tag{28}$$

It remains to substitute the development (23) of $(AX)^*_{uj}$ in (27) and to apply Lemma 1.2.4 to obtain

$$(AX)^*_{ij} = P_{ij} \oplus Q_{ij}(\lambda X^c)^* \tag{29}$$

where

$$\lambda \stackrel{\text{def}}{=} \bigoplus_{i,j} \bigoplus_{k\in F_{ij}} \lambda_{kij} \ .$$

and P_{ij}, Q_{ij} are polynomials. The fact that $\mathcal{S}$ has no divisors of zero is used here in a critical way (in loose terms, this implies that all the λ_{kij} appear in (27)). Then, Proposition 1.1.2 applied to (29) gives the periodicity property for A^n_{ij} with a common rate λ and cyclicity c.

2/*Non primitive case* Let us now consider an irreducible matrix A with cyclicity c, with the following Frobenius normal form (see e.g. [28])

$$A = \begin{bmatrix} \varepsilon & A_1 & \varepsilon & \ldots & \varepsilon \\ \varepsilon & \varepsilon & A_2 & & \vdots \\ \varepsilon & & \varepsilon & \ddots & \\ \vdots & & & \varepsilon & A_{c-1} \\ A_c & \varepsilon & & & \varepsilon \end{bmatrix} .$$

Hence,

$$A^c = \mathrm{diag}(\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_c)$$

where the diagonal blocs

$$\overline{A_1} \stackrel{\text{def}}{=} A_1 \ldots A_c, \ \ \overline{A}_2 \stackrel{\text{def}}{=} A_2A_3 \ldots A_cA_1, \ldots$$

are primitive. The first part of the proof (primitive case) shows that $(\overline{A}_1)^{n+k} = \lambda^k(\overline{A}_1)^n$ for some $k \ge 1$ and n large enough. Since

$$(\overline{A}_2)^n = A_2 \ldots A_c(\overline{A}_1)^{n-1}A_1 \ ,$$

we conclude that $(\overline{A}_2)^n$ is also periodic with the same λ and k. Since an analogous argument applies to the others diagonal blocs of A^c, we are done. ■

1.2.5 Example We show that when A is not irreducible, the sequence $A^n_{ij}, n \in \mathbb{N}$ can admit several distincts rates, so that Corollary 1.2.2 cannot be refined.

$$A = \begin{bmatrix} \varepsilon & e & \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon & -2 & e \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ e \end{bmatrix},$$

$$C = \begin{bmatrix} e & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} \tag{30}$$

Let $s_n = CA^nB$. We have $s_0 = s_1 = \varepsilon$, $\forall n \geq 1, s_{2n+1} = 2n-1, s_{2n} = -4n+4$. This is the merge of two geometric series with distinct rates $\lambda_0 = -4$ et $\lambda_1 = 2$.

1.2.6 Example Strong stabilization cannot be replaced by weak stabilization in Theorem 1.2.3. Consider for instance

$$A = \begin{bmatrix} (e,e) & (e,\varepsilon) \\ (e,\varepsilon) & (e,1) \end{bmatrix} \in (\mathbb{R}_{\max,c}{}^2)^{2\times 2}$$

(see 1.1.8 for the definition of the dioid $\mathbb{R}_{\max,c}{}^2$, which satisfies the weak stabilization but not the strong one). The matrix A is obviously irreducible, but it is plain that the sequence

$$A^n = \begin{bmatrix} (e,e) & (e,\varepsilon) \\ (e,\varepsilon) & (e,n) \end{bmatrix}$$

does not satisfy 1.2.3.

2 Bounds for the Minimal Realization Problem

2.1 Upper Bound via Weak Rank

The *Hankel matrix* associated with $s = \bigoplus_k \langle s, X^k\rangle X^k$ is the $\mathbb{N}\times\mathbb{N}$-matrix:

$$\mathcal{H} = \begin{bmatrix} \langle s,X^0\rangle & \langle s,X\rangle & \langle s,X^2\rangle & \dots & \\ \langle s,X\rangle & \langle s,X^2\rangle & \langle s,X^3\rangle & \dots & \\ \langle s,X^2\rangle & \langle s,X^3\rangle & & & \\ \vdots & \vdots & & & \ddots \end{bmatrix}$$

In the classical theory of realization of rational series over fields [14], the rank of the Hankel matrix provides the minimal dimension of realization. For rational series over dioids, there are some weaker notions of rank which only provide bounds for the minimal realization problem and that we discuss now. Indeed, in the case of dioids, the complete solution of the minimal realization problem remains open and seems almost as difficult as in the case of positive rational series [15].

In the following, we shall speak of moduloids which are defined over dioids in a way similar to modules over rings [34, 16].

2.1.1 Definition (Weak Rank) *The weak dimension of a moduloid is the minimal cardinal of a generating family. The* weak column rank *of a matrix A denoted by $\mathrm{rg}_w A$, is the weak dimension of the moduloid generated by the columns of A.*

Here, we shall just deal with Hankel matrices which are symmetric and we shall not need the notion of weak row rank (defined dually). The notions of rank and linear dependence over dioids have been previously studied by Cuninghame-Green [9, 10], Moller [29] and Wagneur [34]. Minimal generating families can by obtained by standard residuation techniques [4, 5, 9, 10, 16].

Assume that s is the merge of the ultimately geometric series $s^{(0)}, \dots, s^{(k-1)}$, i.e. that (13) holds. Let $\mathcal{H}^{(i)}$ denote the Hankel matrix associated with $s^{(i)}(X^k)$. The following upper bound is a straightforward generalization of a result given by Cuninghame-Green in [10] and valid for the subclass of ultimately geometric series.

2.1.2 Proposition (Upper Bound) *There exists a realization of s of dimension $\sum_{i=1}^{k} \mathrm{rg}_w \mathcal{H}^{(k)}$.*

Hence, by Theorem 1.1.10, this proposition provides an upper bound for the minimal realization of a rational series over a dioid satisfying the stabilization condition (such as $\mathbb{R}_{\max}$).

Proof Recall that if two series u, u' admit two realizations (C, A, B), (C', A', B') of respective size n and n', then, $u \oplus u'$ admit a realization of size $n + n'$, namely

$$C'' = [C, C'], \quad A'' = \begin{bmatrix} A & \varepsilon \\ \varepsilon & A' \end{bmatrix}, \quad B'' = \begin{bmatrix} B \\ B' \end{bmatrix} \ .$$

Hence, it is enough to show the theorem when $s = s^{(i)}(X^k)$. i.e. when s is ultimately geometric, which is the case considered by Cuninghame Green [10]. For the sake of completeness, we recall an alternative realization algorithm which is classical [22] and a bit simpler than that of [10]. Let $\mathcal{H}_{\cdot,i_1}, \ldots, \mathcal{H}_{\cdot,i_r}$ be a minimal generating family of the moduloid spanned by the columns of $\mathcal{H}$ (hence, $r = \mathrm{rg}_w \mathcal{H}$). Thus, there exists a $r \times r$-matrix A such that

$$[\mathcal{H}_{\cdot,i_1+1}, \ldots, \mathcal{H}_{\cdot,i_r+1}] = [\mathcal{H}_{\cdot,i_1}, \ldots, \mathcal{H}_{\cdot,i_r}]A \ .$$

Similarly, there exists $B \in \mathcal{S}^r$ such that

$$\mathcal{H}_{\cdot,0} = [\mathcal{H}_{\cdot,i_1}, \ldots, \mathcal{H}_{\cdot,i_r}]B$$

(we number the columns of the Hankel matrix from 0). It remains to set $C = [\mathcal{H}_{0,i_1}, \ldots, \mathcal{H}_{0,i_r}]$: (C, A, B) is a realization of s of size $\mathrm{rg}_w \mathcal{H}$. We note that this realization is effective in the case of $\mathbb{R}_{\max}$ since A, B can be obtained by residuation of finite matrices[4, 5, 9, 10, 16]. ∎

2.2 Lower Bound via Minor Rank

We next provides a lower bound for the realization problem. We shall need the notion of *symmetrized semiring*, which has been introduced in [30] and more completely studied in [16]. We say that $(\mathcal{S}, \oplus, \otimes, \ominus)$ is a *symmetrized semiring* if $(\mathcal{S}, \oplus, \otimes)$ is a semiring and if $\ominus$ is an unary operator which satisfies the three following properties:

$$\begin{aligned} \ominus(a \oplus b) &= (\ominus a) \oplus (\ominus b) \\ (\ominus a)b &= a(\ominus b) = \ominus(ab) \\ \ominus \ominus a &= a \ . \end{aligned}$$

A map φ is a morphism of symmetrized semiring if it is a morphism of semiring which satisfies

$$\varphi(\ominus x) = \ominus \varphi(x) \ .$$

Rings equipped with the usual minus sign are obvious examples of symmetrized semirings. A natural problem which extends the usual symmetrization of $\mathbb{N}$ by $\mathbb{Z}$ consists in embedding a semiring $\mathcal{S}$ into a symmetrized semiring $\mathcal{S}'$. This problem is studied in [16]. Here, we shall only need the *free symmetrized semiring* $\mathcal{S}^2$, that we define now. We consider $\mathcal{S}^2$ equipped with the following operations

$$\begin{aligned} (a', b') \oplus (a'', b'') &= (a' \oplus a'', b' \oplus b''), \\ (a', b') \otimes (a'', b'') &= (a'a'' \oplus b'b'', a'b'' \oplus a''b'), \\ \ominus(a', a'') &= (a'', a') \ . \end{aligned}$$

The null element is $(\varepsilon,\varepsilon)$ and the unit is (e,ε). We shall as usual write $a \ominus b$ for $a \oplus \ominus b$. We have an injective morphism of semirings

$$\mathcal{S} \to \mathcal{S}^2, \ \ x \mapsto (x,\varepsilon) \ ,$$

hence, $\mathcal{S}^2$ provides a symmetrization of $\mathcal{S}$, and we shall identify $\mathcal{S}$ to the subsemiring of $\mathcal{S}^2$ composed of the elements of the form (x,ε), writing $x=(x,\varepsilon)$. With this convention, an element x of $\mathcal{S}^2$ admits a unique decomposition

$$x = x' \ominus x'', \ \ x',x'' \in \mathcal{S} \ . \tag{31}$$

We define the *determinant* of a matrix A with entries in a symmetrized semiring by

$$\det A = \bigoplus_{\sigma} \operatorname{sgn}\sigma \bigotimes_{i=1}^{n} A_{i\sigma(i)} \tag{32}$$

where the sum is taken over the permutations of $\{1,\ldots,n\}$ and $\operatorname{sgn}\sigma = e$ if σ is even and $\ominus e$ is σ is odd.

We next extend some well known properties of determinants to symmetrized semirings. We first define the *balance* relation ∇ -which will replace to some extent the equality relation-:

$$(a',a'') \nabla (b',b'') \iff a' \oplus b'' = a'' \oplus b' \ .$$

Observe that ∇ is *not* transitive in $\mathbb{R}^2_{\max}$. Now, let $\mathcal{S}$ be a commutative semiring, and let φ denote the unique morphism of symmetrized semirings $(\mathbb{N}[X_1,\ldots,X_k])^2 \to (\mathcal{S}[X_1,\ldots,X_k])^2$ such that $\forall i, \varphi(X_i) = X_i$. The following transfer principle states that combinatorial identities valid in commutative rings involving $+,\times,-$ also hold in symmetrized commutative semirings, provided that equalities are replaced by ∇ . This is just a formalization of a fact already noticed by Reutenauer and Straubing [32].

2.2.1 Transfer Principle *Let $P^+,P^-,Q^+,Q^- \in \mathbb{N}[X_1,\ldots,X_k]$. Assume that*

$$P^+ - P^- = Q^+ - Q^-, \ \text{ holds in } \ \mathbb{Z}[X_1,\ldots,X_k] \ .$$

Then,

$$\varphi(P^+,P^-) \nabla \varphi(Q^+,Q^-) \ \text{ holds in } \ \mathcal{S}^2[X_1,\ldots,X_k] \ .$$

Proof immediate from the definition of ∇ . ■

This allows translating the well known Binet-Cauchy formula. Let us denote by $A_{[I|J]}$ the submatrix $(A_{ij})_{i\in I,\, j\in J}$.

2.2.2 Proposition *Let $\mathcal{S}$ be an arbitrary commutative semiring. Let $A \in \mathcal{S}^{n\times r}$, $B \in \mathcal{S}^{r\times p}$, $I \subset \{1,\ldots,n\}$, $J \subset \{1,\ldots,p\}$ with $\#I = \#J = k$. The following identity holds in $\mathcal{S}^2$:*

$$\det(AB)_{[I|J]} \nabla \bigoplus_{K} \det A_{[I|K]} . \det B_{[K|J]} \ , \tag{33}$$

where the sum is taken over all the subsets $K \subset \{1,\ldots,r\}$ of cardinal k (this sum is empty, conventionally equal to ε if $k > r$).

Proof Let us first consider the entries a_{ij}, b_{kl} of the matrices as indeterminates. By applying the transfer principle to the classical Binet-Cauchy formula in $\mathbb{Z}[a_{ij}, b_{ij}]$, we see that the identity holds formally in the symmetrized semiring $(\mathcal{S}[a_{ij}, b_{kl}])^2$. Hence it is true for all values of a_{ij}, b_{kl}. ∎

It is also possible to provide a combinatorial proof along the lines of Zeilberger [35].

We say that an element $x \in \mathcal{S}^2$ is *balanced* if $x \nabla \varepsilon$, unbalanced otherwise. Then, we define the minor rank of a matrix.

2.2.3 Definition (Minor Rank) *The* minor rank *of a matrix is by definition the maximal size of a square submatrix with unbalanced determinant.*

2.2.4 Theorem (Lower Bound) *Let $\mathcal{S}$ be an arbitrary commutative semiring and $s \in \mathcal{S}[[X]]$. The dimension of any realization of s is at least equal to the minor rank of its Hankel matrix.*

Observe that in the case of fields, this bound coincides with the minimal dimension of realization [14].

Proof This follows from $\mathcal{H} = \mathcal{O}\mathcal{C}$, where $\mathcal{O} = [C, CA, CA^2, \ldots]^{\mathrm{T}}$ and $\mathcal{C} = [B, AB, A^2B, \ldots]$ are the usual observability and controllability matrices. Consider a minor extracted from $\mathcal{H}$, $\det \mathcal{H}_{[I|J]}$. The Binet-Cauchy formula states that

$$\det \mathcal{H}_{[I|J]} \nabla \bigoplus_{K} \det \mathcal{O}_{[I|K]} \det \mathcal{C}_{[K|J]} \ .$$

This sum is empty as soon as the cardinal of I is greater than the size p of the realization. Hence, $\det \mathcal{H}_{[I|J]} \nabla \varepsilon$ for all $I \times J$ submatrix of $\mathcal{H}$ of size $> p$. ∎

2.2.5 Remark This bound is easily extended to rational series with several noncommuting indeterminates. Let $\mathcal{S}$ be a commutative semiring, Σ^* the free monoid over a finite alphabet Σ and consider a series $s \in \mathcal{S}\langle\langle\Sigma\rangle\rangle$. The Hankel matrix of s [14] is the $\Sigma^* \times \Sigma^*$ matrix $\mathcal{H}$: $\mathcal{H}_{u,v} \stackrel{\text{def}}{=} \langle s, uv \rangle$. Let λ, μ, γ be a linear representation of dimension p of s, i.e. $\lambda \in \mathcal{S}^{1\times p}, \gamma \in \mathcal{S}^{p\times 1}$, μ is a morphism $\Sigma^* \to \mathcal{S}^{p\times p}$, and $\langle s, w \rangle = \lambda\mu(w)\gamma$. We have $\mathcal{H} = \mathcal{O}\mathcal{C}$ with $\mathcal{O} \in \mathcal{S}^{\Sigma^* \times p}$, $\mathcal{O}_u = \lambda\mu(u)$, and $\mathcal{C} \in \mathcal{S}^{p\times\Sigma^*}$, $\mathcal{C}_v = \mu(v)\gamma$. The same argument shows that the minor rank of $\mathcal{H}$ is a lower bound for the dimension p of the linear representation of s.

3 Some Examples

We next exhibit a few examples which should made it clear when the lower and upper bounds for the minimal realization problem are accurate.

3.1 An Example where Weak Rank and Minor Rank Coincide

Consider the realization

$$A = \begin{bmatrix} 4 & \varepsilon & \varepsilon \\ \varepsilon & 5 & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \varepsilon \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \varepsilon & 0 \end{bmatrix}$$

with truncated Hankel matrix

$$\mathcal{H}_{[0..6|0..6]} = \begin{bmatrix} 0 & 4 & 8 & 12 & 16 & 20 & 25 \\ 4 & 8 & 12 & 16 & 20 & 25 & 30 \\ 8 & 12 & 16 & 20 & 25 & 30 & 35 \\ 12 & 16 & 20 & 25 & 30 & 35 & 40 \\ 16 & 20 & 25 & 30 & 35 & 40 & 45 \\ 20 & 25 & 30 & 35 & 40 & 45 & 50 \\ 25 & 30 & 35 & 40 & 45 & 50 & 55 \end{bmatrix}$$

(the other columns of $\mathcal{H}$ are proportional to the the 6-th one). We have

$$\det \mathcal{H}_{[12|56]} = \det \begin{bmatrix} 16 & 20 \\ 20 & 25 \end{bmatrix} = 16 \otimes 25 \ominus 20 \otimes 20 = 41 \ominus 40 = (41, 40) \not\nabla \varepsilon \ .$$

Hence, the dimension of realization is at least 2. Indeed, let us apply the realization algorithm sketched in the Proof of Proposition 2.1.2. It is easily seen that the columns of indices 0 and 5 are a minimal generating family of the moduloid of columns of $\mathcal{H}$. Hence, we obtain the following 2-dimensional realization

$$A' = \begin{bmatrix} 4 & 25 \\ -20 & 5 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}$$

$$C' = \begin{bmatrix} 0 & 20 \end{bmatrix}$$

The lower bound shows that it is minimal.

3.2 A Series with Large Weak Rank but Small Minimal Realization

Consider the following triple

$$A = \begin{bmatrix} 5 & \varepsilon & 0 \\ \varepsilon & 4 & 0 \\ \varepsilon & \varepsilon & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 4 & 6 \end{bmatrix}$$

with truncated Hankel matrix:

$$\mathcal{H}_{[0..5|0..5]} = \begin{bmatrix} 6 & 9 & 12 & 16 & 20 & 25 \\ 9 & 12 & 16 & 20 & 25 & 30 \\ 12 & 16 & 20 & 25 & 30 & 35 \\ 16 & 20 & 25 & 30 & 35 & 40 \\ 20 & 25 & 30 & 35 & 40 & 45 \\ 25 & 30 & 35 & 40 & 45 & 50 \end{bmatrix}$$

Since $\det \mathcal{H}_{[0,1,2|0,2,5]} = 57 \ominus 56 \not\nabla \varepsilon$, $\mathcal{H}$ has minor rank 3 and (A, B, C) is a minimal realization. However, the above mentioned residuation techniques show that the columns of indices 0,1,2,4 are a minimal generating family of the set of columns of the Hankel matrix. Hence, the upper bound via weak rank is 4 which is greater to actual minimal dimension of realization. Indeed, the situation can be much worse. Let us consider more generally the following matrix:

$$C = \begin{bmatrix} 0 & r & s \end{bmatrix} \ .$$

Let $H \stackrel{\text{def}}{=} C(AX)^*B$. We have

$$H = (5X)^* \oplus r(4X)^* \oplus s(3X)^* \ .$$

3.2.1 Proposition *The weak rank of the Hankel matrix of H can be arbitrarily large for properly chosen r, s.*

First, we have

$$\langle H, X^N\rangle = \max(5\times N, r+4\times N, s+2\times N) = N^5 \oplus rN^4 \oplus sN^3 = N^3(N\oplus r\oplus\sqrt{s})(N\oplus(\frac{s}{r}\wedge\sqrt{s}))$$

(recall that $\sqrt{s}$ in the $(\max,+)$ algebra stands for $s/2$ in the usual algebra). Let us assume $r^2 > s$, then, the map $k \mapsto \langle H, X^k\rangle = \mathcal{H}_{k,0}$ is the max of the three affine functions drawn in Figure 1, with the following two distinct corners:

$$n_1 = r > n_2 = \frac{s}{r} \ .$$

Let us further suppose that $s, r \in \mathbb{N}$ and $s > r$ which implies that $\frac{s}{r} = s - r \in \mathbb{N}$. Then

$$\mathcal{H}_{ni} = (ni)^3(ni\oplus r)(ni\oplus\frac{s}{r}) \ . \tag{34}$$

3.2.2 Lemma *Under the foregoing assumptions, if $1\otimes\frac{s}{r} \prec r$, the columns $\mathcal{H}_{\cdot,0},\ldots,\mathcal{H}_{\cdot,\frac{s}{r}}$ belong (up to a scaling) to any generating family of the moduloid of columns of $\mathcal{H}$.*

This result might seem unnatural to the reader who is not familiar with rank theory in moduloids: it is important to note that minimal generating families of moduloids of finite type over $\mathbb{R}_{\max}$ are unique (up to a permutation and a scaling) [29, 34], due to the fact that the only invertible matrices are products of permutation and diagonal matrices. Thus, the elements of minimal generating families are akin to the extremal directions of (conventional) convex cones. The above lemma exhibits a subset of $1\otimes\frac{s}{r} = s - r + 1$ "extremal columns" of the Hankel matrix, which implies that the weak column rank of $\mathcal{H}$ which is at least $s - r + 1$ can be arbitrarily large with respect to the minimal dimension of realization (at most 3). It should be intuitively clear by looking at Figure 1

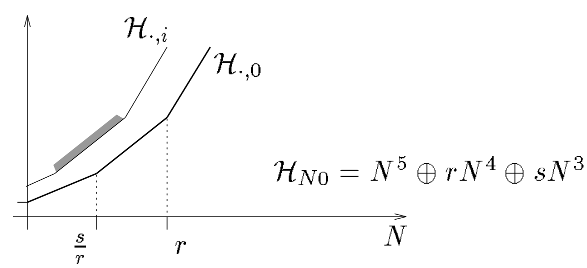

Figure 1: A series with large weak rank

that the columns $\mathcal{H}_{\cdot,i}$ (for $0 \le i \le s - r$) are extremal, i.e. that there does not exist a relation of the form

$$\mathcal{H}_{\cdot,i} = \bigoplus_{j\in J} \lambda_j \mathcal{H}_{\cdot,j} \ , \tag{35}$$

where J is a finite subset of $\mathbb{N}$ which does not contain i. Indeed, λ_j must be such that the map associated to the column $\lambda_j\mathcal{H}_{\cdot,j}$ lies below $\mathcal{H}_{\cdot,i}$. It should be geometrically clear that the part of the graph of $\mathcal{H}_{\cdot,i}$ between $\frac{s}{r} - i$ and $r - i$ (shaded part) will never be attained by the max at the right hand of (35). A more precise (but technical) proof of this Lemma is provided in Appendix III.

3.3 A Rational Series with Infinite Weak Rank

The weak rank of the Hankel matrix of a rational series which is not ultimately geometric can be infinite. This is why we have to consider all the Hankel matrices $\mathcal{H}^{(0)}, \ldots, \mathcal{H}^{(k-1)}$ in Proposition 2.1.2. For instance, let us consider the series

$$s = (X^2)^* \oplus 1X((1X)^2)^* \in \mathbb{R}_{\max}[[X]] \ ,$$

i.e.

$$\langle s, X^i \rangle = \begin{cases} e & \text{if } i \text{ even} \\ 1^i = i \times 1 = i & \text{if } i \text{ odd.} \end{cases}$$

We claim that *the moduloid generated by the columns of the Hankel matrix of s does not admit a finite generating family.* Indeed, let $\{\mathcal{H}_{\cdot,i_1}, \ldots, \mathcal{H}_{\cdot,i_k}\}$ be a finite generating family of the moduloid of columns of $\mathcal{H}$. We have for all $i \in \mathbb{N}$ a linear combination of the form

$$\mathcal{H}_{\cdot,i} = \bigoplus_{l=1}^{k} \lambda_{il} \mathcal{H}_{\cdot,i_l} \ . \tag{36}$$

(i) *If i an i_l are not of the same parity, then $\lambda_{il} = \varepsilon$.* This follows from

$$\lambda_{il} \leq \bigwedge_{j\in\mathbb{N}} \frac{\mathcal{H}_{j,i}}{\mathcal{H}_{j,i_l}} = \left(\bigwedge_{j\in\mathbb{N}\cap(i+2\mathbb{Z})} \frac{e}{1^{j+i_l}} \right) \wedge \left(\bigwedge_{j\in\mathbb{N}\cap(i+2\mathbb{Z}+1)} \frac{1^{j+i}}{e} \right) = \varepsilon \ .$$

(ii) *We have for all i and l, $\lambda_{il} \leq e$.* Indeed, from (i), we are reduced to the case where i and i_l have the same parity. Then

$$\lambda_{il} \leq \frac{\mathcal{H}_{ii}}{\mathcal{H}_{ii_k}} = \frac{e}{e} = e \ .$$

These two remarks together with (36) imply that the first row of $\mathcal{H}$ is bounded above: a contradiction.

3.4 An Irrational Series with Finite Minor Rank

3.4.1 Proposition *Consider the series $s = \bigoplus_k s_k X^k$, with*

$$s_k = \begin{cases} \varepsilon & \text{if } \exists p \in \mathbb{N},\ \ k = 3^p \\ e & \text{otherwise} \end{cases}$$

Although s is not realizable, the minor rank of its Hankel matrix is finite.

Proof Plainly, s_k is not a merge of ultimately geometric series, hence it is not realizable. We show that the large minors taken from the Hankel matrix are equal to $e \ominus e$ (hence, they are balanced). Let $\mathbb{B} = \{\varepsilon, e\}$ denote the boolean dioid, which is a subdioid of $\mathbb{R}_{\max}$. The key point in the proof is the following observation which shows that boolean minors with too many e entries are balanced.

3.4.2 Proposition *Let $M \in \mathbb{B}^{n\times n}$. If $\det M \ \not\nabla\ \varepsilon$, then, at least $\frac{n(n-1)}{2}$ entries of M are equal to ε.*

Proof For a 2×2 matrix, this is clear. After multiplying M by a permutation matrix (which changes the sign of $\det M$ but not the fact that it is balanced or not), we may assume that $\bigotimes_i M_{ii} = e$. If the M_{ij} entry is nonzero, then, then the M_{ji} entry must be zero (otherwise, we apply the proposition already proved for 2×2 matrices, and we get by minor expansion $\det M = \det M[ij|ij]e \oplus \ldots = e \ominus e \oplus \ldots = e \ominus e$). Thus, at most one half of the outdiagonal entries of M can be nonzero. ■

It should be intuitively clear that since s_k takes asymptotically very few zero values, due to this proposition, the large minors of the Hankel matrix are balanced. The precise proof seems less immediate, and the proof that we offer relies on the following perhaps artificial trick.

3.4.3 Definition *We say that the matrix A has* no lower triangles *if the following situation does not occur*

$$A_{ij} = \varepsilon, \quad A_{lk} = \varepsilon, \quad A_{\max(i,l),\max(j,k)} = \varepsilon \quad \text{pour } i \neq l \text{ et } j \neq k.$$

This is illustrated by Figure 2:

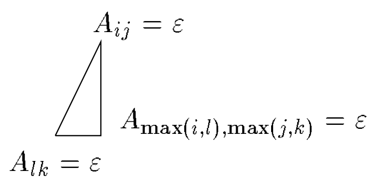

Figure 2: Lower triangle

3.4.4 Lemma *The Hankel matrix of the series s has no lower triangles.*

Proof If $i + j = 3^p$, $l + k = 3^q$, we have $\max(i,l) + \max(j,k) \leq 2 \times 3^{\max(p,q)}$. If $\mathcal{H}$ admits a lower triangle, we must have $\max(i,l) + \max(j,k) = 3^r$ with $r > \max(p,q)$, hence $3^r \leq 2 \times 3^{\max(p,q)}$: a contradiction. ■

It remains to show that a matrix without lower triangles cannot have too many zeros, which together with Proposition 3.4.1 will show that the minor rank of $\mathcal{H}$ is finite. Indeed, we have the following more precise result.

3.4.5 Proposition *Let $C(n,p)$ denote the maximal number of zeros of a $n \times p$ matrix without lower triangles. We have*

$$C(n,p) = n + p - 1 \ .$$

Proof First, $C(n,1) = n$, $C(p,q) = C(q,p)$. Let r denote the number of zeros of the last column of A. We have the following dynamic programming equation:

$$C(n,p) = \max_{1 \leq r \leq n} \left[C(n - r + 1, p - 1) + r \right] \ . \tag{37}$$

Indeed, let us assume that the zeros on the last column of A are in position $(i_1,p),(i_2,p),\dots,(i_r,p)$ with $i_1<i_2<\dots<i_r$, i.e.

$$A=\begin{array}{c} \\ i_1 \\ i_2 \\ i_r \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & \dots & p-1 & p \end{array} \\ \left[\begin{array}{cccc} & & & \varepsilon \\ + & \dots & + & \varepsilon \\ + & \dots & + & \varepsilon \end{array}\right] \end{array}.$$

Then, the entries that are both on the $p-1$ first columns and on the rows $i_2,\dots i_r$ of A are necessarily non zero (these entries are written "+"). Hence, if r is chosen, we are reduced to maximizing the number of zero entries of the submatrix of size $(n-r+1)\times(p-1)$, $A(i_2\dots i_r|p)$ (that is A without the rows marked $+$ and the last column). This gives the induction (37) from which Proposition 3.4.5 easily follows. ∎

Thus, the number of zeros of an $n\times n$ minor taken from $\mathcal{H}$ is at most $C(n,n)=2n-1$. Since $\frac{n(n-1)}{2}>2n-1$ (for $n\geq 5$), Proposition 3.4.2 shows that all the minors of $\mathcal{H}$ of size greater than 5 are balanced, hence, the minor rank of $\mathcal{H}$ is finite (at most 4). ∎

Appendix

I Effective Computation of Determinants

In this appendix, we provide an $O(n^3)$ algorithm to decide if the determinant of an $n\times n$ matrix A with entries in $\mathbb{R}_{\max}$ is balanced. Another specific algorithm should be provided to compute the rank of matrices more efficiently than by minors inspection, but it is beyond the scope of this paper.

Let us define for $a=(a',a'')\in\mathbb{R}^2_{\max}$, $|a|=a'\oplus a''$. Then,

$$\operatorname{perm} A \stackrel{\text{def}}{=} \bigoplus_{\sigma}\bigotimes_{i} A_{i\sigma(i)} = |\det A|$$

is the well known permanent function. As already noticed by Butkovič and Cuninghame-Green [6], computing perm A is equivalent to a standard assignment problem, i.e. with conventional notation

$$\operatorname{perm} A = \max_{\sigma}\sum_{i} A_{i\sigma(i)}$$

Hence, computing perm A is $O(n^3)$. Since $|\det A| = \operatorname{perm} A$, the case $\operatorname{perm} A=\varepsilon$ which implies that $\det A=\varepsilon$ is trivial, and we shall assume that $\operatorname{perm} A\neq\varepsilon$. Hence, there is a permutation σ such that $\bigotimes_i A_{i\sigma(i)}=\operatorname{perm} A\neq\varepsilon$. Consider the matrix

$$C_{ij}=\begin{cases} A_{i\sigma(i)} & \text{if } j=\sigma(i) \\ \varepsilon & \text{otherwise}\end{cases}$$

and let

$$B=C^{-1}A \tag{38}$$

Since $\det B=\det C^{-1}\det A$, we are reduced to the following

Canonical Form:

$$B\geq \mathrm{Id},\quad \operatorname{perm} B=e \tag{39}$$

Given a cycle $c = (i_1, \ldots, i_k)$, the weight of c with respect to the matrix B is by definition

$$w_B(c) = B_{i_1 i_2} \ldots B_{i_k i_1} \enspace .$$

We shall write $w(c)$ instead of $w_B(c)$ when the matrix will be clear from the context.

I.1 Lemma *For a matrix B satisfying (39), for all circuit c, $w(c) \leq e$.*

Proof Let us first notice that all the diagonal elements of B are equal to e, for

$$\forall i, \quad e \leq B_{ii} \leq B_{ii} \bigotimes_{j \neq i} B_{jj} \leq \operatorname{perm} B = e \enspace .$$

Next, consider a circuit c and let σ be the cyclic permutation associated with c. We have, since $B_{ii} = e$ for all i,

$$\bigotimes_i B_{i\sigma(i)} = w(c) \leq \operatorname{perm} B = e \enspace .$$

■

Let us denote by F the "out-diagonal" part of B, i.e.

$$B = \mathrm{Id} \oplus F, \quad F_{ij} = \begin{cases} B_{ij} & \text{if } i \neq j \\ \varepsilon & \text{if } i = j. \end{cases} \enspace . \tag{40}$$

The following notation is standard [20, 9]:

$$A^+ = A \oplus A^2 \oplus A^3 \oplus \ldots$$

From Lemma I.1, we get that B^+ converges (this is a well known theorem about stars of matrices in the $(\max, +)$ algebra [20], Chapter 3,§2.3).

I.2 Proposition *Let $B = \mathrm{Id} \oplus F$ as in (40), with $\operatorname{perm} B = e$. The following assertions are equivalent*

1. $\det B \nabla \varepsilon$
2. $\det B = e \ominus e$
3. *There exists a circuit of F of weight e and even length.*
4. $\operatorname{tr}(F^2)^+ = e$.

The following algorithm is an immediate consequence of this proposition.

I.3 Algorithm *Let $A \in \mathbb{R}_{\max}^{n\times n}$. We decide whether* $\det A$ *is balanced.*

1. *Compute* $\operatorname{perm} A$ *using an assignment algorithm (e.g. [20], Hungarish Algorithm (p. 157)). If* $\operatorname{perm} A = \varepsilon$, *then* $\det A = \varepsilon$ *and we are done. Otherwise, we get a permutation σ such that*

$$\operatorname{perm} A = \bigotimes_i A_{i\sigma(i)} \neq \varepsilon \enspace .$$

2. *Define B and F by (38,40).*
3. *If* $\operatorname{tr}(F^2)^+ = e$, *then* $\det A \nabla \varepsilon$, *otherwise* $\det A \not\nabla \varepsilon$.

More precisely, the algorithm shows that if $\mathrm{tr}(F^2)^+ = e$,

$$\det A = \mathrm{perm}\, A \ominus \mathrm{perm}\, A$$

and if $\mathrm{tr}(F^2)^+ \neq e$,

$$\det A = \mathrm{sgn}\,(\sigma) \otimes (\mathrm{perm}\, A \ominus t)$$

for some $t < \mathrm{perm}\, A$. Indeed, this result would write more simply

$$\det A = \mathrm{sgn}\,(\sigma)(e \ominus \mathrm{tr}(F^2)^+)\mathrm{perm}\, A$$

in the symmetrized semiring $\mathbb{S}_{\max}$ [30, 16, 1].

Since the computation of $(F^2)^+$ using the Jordan algorithm is $O(n^3)$ (see e.g. [20], Chapter 3, §4) and the standard assignment algorithms are $O(n^3)$, checking if $\det A \;\nabla\; \varepsilon$ is $O(n^3)$.

Proof of the Proposition. (1)$\Leftrightarrow$(2) follows immediately from $|\det B| = \mathrm{perm}\, B = e$ and the definition of ∇ . (3)$\Leftrightarrow$(4) follows from the fact that $\mathrm{tr}(A^+)$ is equal to the sum of the weights of the circuits of A.
(3) $\Rightarrow$ (2). Write $\det B = x' \ominus x''$ as in (31). Since $B \geq \mathrm{Id}$, $\det B \geq \det \mathrm{Id} = e$, hence, $x' \geq e$. Since $|\det B| = x' \oplus x'' = \mathrm{perm}\, B = e$, we get $x' \leq e, x'' \leq e$, hence $x' = e$. Let $c = (i_1, \ldots, i_{2p})$ be a circuit of length $2p$ and weight e, and introduce the permutation σ such that $\sigma(j) = j$ if $j \notin \{i_1, \ldots, i_{2p}\}$ and $\sigma(i_k) = i_{k+1}$ (convention $i_{2p+1} = i_1$). Clearly, $\mathrm{sgn}\,(\sigma) = \ominus e$. Hence,

$$\det B \geq \ominus \bigotimes_i B_{i\sigma(i)} = \ominus \bigotimes_{l=1}^{2p} B_{i_l i_{l+1}} = \ominus \bigotimes_{l=1}^{2p} F_{i_l i_{l+1}} = \ominus e$$

for $B_{j\sigma(j)} = B_{jj} = e$ if $j \notin \{i_1, \ldots, i_{2p}\}$ and $B_{j\sigma(j)} = F_{j\sigma(j)}$ otherwise. This implies that $x'' \geq e$, hence $x'' = e$ and $\det B = e \ominus e$.
(2) $\Rightarrow$ (4). Consider the decomposition of a permutation $\sigma = c_1 \ldots c_k$ as a product of disjoint cycles of respective lengths $l_1, \ldots, l_k$, and define the weight of σ by $w(\sigma) = w(c_1) \ldots w(c_k)$. Since $\det B = \bigoplus_\sigma \mathrm{sgn}\,(\sigma) w(\sigma) = e \ominus e$, there exists a permutation σ such that $\mathrm{sgn}\,(\sigma) w(\sigma) = \ominus e$. Since $\mathrm{sgn}\,(\sigma) = \ominus e$, σ admits at least one cycle of even length. Moreover,

$$e = w(\sigma) = w(c_1) \ldots w(c_k)$$

with $w(c_i) \leq e$ for all i implies that $w(c_1) = \ldots = w(c_k) = e$. Hence, there is a circuit of F with even length and weight e. ■

I.4 Example To illustrate this algorithm, let us consider the matrix

$$A = \begin{bmatrix} e & 1 & \varepsilon & \varepsilon \\ \varepsilon & e & 1 & \varepsilon \\ \varepsilon & \varepsilon & e & 1 \\ -3 & \varepsilon & \varepsilon & e \end{bmatrix} .$$

Since there are only two permutations with non ε weight (namely, Id and $(1, 2, 3, 4)$, with weight e and opposite sign), it is immediate that $\det A = e \ominus e$. Let us check this by the algorithm I.2,(4). Here, we have $\mathrm{perm}\, A = e$ and $A = \mathrm{Id} \oplus F$ with

$$F = \begin{bmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \\ -3 & \varepsilon & \varepsilon & \varepsilon \end{bmatrix} .$$

By definition, $(F^2)^+$ is the limit (reached in a finite time) of the sequence $X_1 = F^2, X_{n+1} = F^2 X_n \oplus F^2$. We have

$$X_2 = \begin{bmatrix} 0 & \varepsilon & 2 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 2 \\ -2 & \varepsilon & 0 & \varepsilon \\ \varepsilon & -2 & \varepsilon & 0 \end{bmatrix}, \; X_3 = X_2$$

Hence, $(F^2)^+ = X_2$ and $\mathrm{tr}(F^2)^+ = e$, which yields a second verification that $\det A \mathrel{\nabla} \varepsilon$ via I.2,(4).

I.5 Remark It should also be possible to compute determinants in another symmetrized semiring, namely the semiring $\mathbb{S}_{\mathbf{max}}$ studied in [30, 16]. Some additional simplification rules hold in $\mathbb{S}_{\mathbf{max}}$, e.g. $3 \ominus 2 = 3$. The adaptation of the above algorithm to $\mathbb{S}_{\mathbf{max}}$ is straightforward: the resulting computations in $\mathbb{S}_{\mathbf{max}}$ are essentially equivalent although a bit simpler due to the simplification rules.

II Equivalence of the Different Possible Definitions of Rational Series

We briefly discuss here another possible convention for the definition of rational series (used e.g. in [1]), and show that it is essentially equivalent to the one used in this paper. In the introduction, following the tradition [3], we only defined s^* when s has zero constant coefficients (Axiom (S)). This definition which is the simplest one works in general semirings. In the case of dioids [1], it is possible to define s^* for more general series due to the underlying ordered structure[2]. We briefly show that this alternative convention provides the same class of rational series under mild conditions.

Let us recall that a dioid is canonically ordered by

$$a \leq b \iff a \oplus b = b \ . \tag{41}$$

Let $X \subset \mathcal{S}$. We say that the infinite sum

$$\bigoplus_{x \in X} x$$

is well defined if X admits a least upper bound. Then, we set

$$\bigoplus_{x \in X} x \stackrel{\text{def}}{=} \sup X \ . \tag{42}$$

The dioid of rational series has been defined as the closure of the dioid of polynomials for the following operations: sum and product of series and star of a series with zero constant coefficient (Axiom (S)). Since s^* exists under more general conditions, we define an a priori more general dioid of formal series Rat' as follows:

II.1 Definition Rat' *is the least set of formal series containing polynomials and such that*

$\mathrm{Rat}' \oplus \mathrm{Rat}' \subset \mathrm{Rat}'$
$\mathrm{Rat}' \otimes \mathrm{Rat}' \subset \mathrm{Rat}'$
(S′) If $s \in \mathrm{Rat}'$ and $s^* = \bigoplus_k s^k$ is well defined according to (42), then $s^* \in \mathrm{Rat}'$.

[2]This amounts to replacing the discrete ultrametric topology on $\mathcal{S}[[X]]$ (see [3]) by some order topology [19].

Indeed, we just have replaced (S) by (S') in Definition 0.0.1. We next show that under natural assumptions, the two conventions are equivalent: $\mathrm{Rat} = \mathrm{Rat}'$.

More precisely, we shall assume that the following infinite distributivity property holds

$$\text{If } X \subset \mathcal{S} \text{ and } \bigoplus_{x\in X} x \text{ exists, then } \forall y \in \mathcal{S},\ \bigoplus_{x\in X} xy \text{ exists and } \bigoplus_{x\in X} xy = (\bigoplus_{x\in X} x)y \ . \tag{43}$$

We have:

II.2 Proposition *Let $\mathcal{S}$ be a commutative dioid which satisfies the infinite distributivity (43). Then* $\mathrm{Rat} = \mathrm{Rat}'$.

Proof Clearly, $\mathrm{Rat} \subset \mathrm{Rat}'$. We show that $\mathrm{Rat}' = \mathrm{Rat}$. It is enough to check that if $s \in \mathrm{Rat}$ and if the least upper bound $s^* = \bigoplus_{k\in\mathbb{N}} s^k$ exists, then $s^* \in \mathrm{Rat}$. Let us define for an arbitrary series t, $t^{[p]} \stackrel{\text{def}}{=} \bigoplus_{k\leq p} t^k$. A simple computation shows that $\langle s^{[p]}, X^0\rangle = \langle x, X^0\rangle^{[p]}$ hence, the existence of s^* implies the existence in $\mathcal{S}$ of

$$\langle s, X^0\rangle^* = \langle s^*, X^0\rangle \ .$$

Setting $s' = \bigoplus_{k\geq 1}\langle s, X^k\rangle X^k$, we have $s = \langle s, X^0\rangle \oplus s'$. Under the infinite distributivity assumption (43), the identity $(a\oplus b)^* = a^*b^*$ holds, hence, $s^* = \langle s, X^0\rangle^* s'^*$ which belongs to Rat since $\langle s, X^0\rangle^*$ is a scalar and $s' \in \mathrm{Rat}$ (for $\langle s', X^0\rangle = \varepsilon$). ■

III Proof of Lemma 3.2.2

Let us assume a linear combination of the form (35). Observe that $\langle H, X^k\rangle = s \otimes k^3$ for $k \leq s/r$.

a/ *If $j < i$, then $\lambda_j \leq (\frac{i}{j})^3$.* Indeed, we get from (34) and

$$\mathcal{H}_{ni} \geq \lambda_j \mathcal{H}_{nj} \quad \text{with } n = 0 \tag{44}$$

that

$$\mathcal{H}_{0i} = i^3 s \geq \lambda_j \mathcal{H}_{0j} = \lambda_j j^3 s,$$

which shows a/.

b/ *If $j > i$, then $\lambda_j \leq (\frac{i}{j})^5$.* This follows from

$$\mathcal{H}_{ni} = (ni)^5 \geq \lambda_j \mathcal{H}_{nj} = \lambda_j (nj)^5,$$

which holds for n large enough.

c/ *Let us choose an integer k such that $\frac{s}{r} < ki < r$. We claim that $\bigoplus_{j\in J}\lambda_j\mathcal{H}_{kj} < \mathcal{H}_{ki}$.*
In the case where $j < i$, we get via a/

$$\begin{aligned}\lambda_j\mathcal{H}_{kj} \leq (\frac{i}{j})^3\mathcal{H}_{kj} &= (\frac{i}{j})^3(kj)^3(kj\oplus r)(kj\oplus\frac{s}{r})\\ &= i^3k^3r(kj\oplus\frac{s}{r}) = r(ki)^4\frac{(kj\oplus\frac{s}{r})}{ki} < r(ki)^4 = \mathcal{H}_{ki} \ .\end{aligned}$$

If $j > i$, we obtain by b/

$$\begin{aligned}\lambda_j\mathcal{H}_{kj} \leq (\frac{i}{j})^5\mathcal{H}_{kj} &= (\frac{i}{j})^5(kj)^3(kj\oplus r)(kj\oplus\frac{s}{r})\\ &= \frac{i^5}{j^2}k^3(kj\oplus\frac{s}{r})kj = r(ki)^4(\frac{ki}{r}\oplus\frac{i}{j}) < r(ki)^4 = \mathcal{H}_{ki} \ .\end{aligned}$$

This concludes the proof of c/ and of the Lemma. ■

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